Writing $h = b - a$, an approximate solution to this equation is

$$kh \sim q\pi + (h^2/8q\pi ab)[8\beta + 4(n-1)^2 - 1] \qquad q = 1,2,3, \ldots \quad (16)$$

For an infinite plate of thickness h, $a,b \to \infty$ and $kh = q\pi$, $q = 1,2,3 \ldots$, which would be the plane-wave solution for an orthotropic plate.

The forementioned approximate formulas (14) and (16) have been checked by comparisons with the exact solutions, and the results are found to be accurate within 5% for values of δ up to $\frac{1}{2}$.

An Error Analysis in the Digital Computation of the Autocorrelation Function

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Introduction

LET $f_1(t)$ be a function of time which is defined and continuous and satisfies all hypotheses of the ergodic theorem[4] on $-\infty < t < +\infty$. Choose a finite record, say $-T \leqslant t \leqslant +T$, and let $f_1(t)$ satisfy the quasi-ergodic hypothesis[3] on $-T$ to $+T$. The purpose of this paper is to deduce a relationship between error ϵ and maximum time lag τ_m in the digital computation of the autocorrelation function of $f_1(t)$ over $-T \leqslant t \leqslant T$. It thus will be demonstrated that the maximum time lag τ_m should not exceed 5 to 10% of the total time record, as suggested by Blackman and Tukey.[1] Details of the analysis that follows can be found in Ref. 2.

Analysis of the Problem

The unnormalized autocorrelation function of $f_1(t)$ can be defined by the equation

$$A_{11}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(t) f_1(t+\tau)\, dt \qquad (1)$$

where τ is the time lag.

If only a finite record is available, then $A_{11}(\tau)$ is approximated by

$$A_{11}(\tau) \doteq \frac{1}{2T} \int_{-T}^{T} f_1(t) f_1(t+\tau)\, dt \qquad (2)$$

A typical graph of $f_1(t)$ and $f_1(t+\tau)$ is shown in Fig. 1.

Define the integrand of Eq. (2) as

$$F(t) = f_1(t) f_1(t+\tau) \qquad (3)$$

A typical graph of $F(t)$ is shown in Fig. 2.

To approximate digitally $A_{11}(\tau)$ from Eq. (2), divide the interval $-T \leqslant t \leqslant T - \tau$ into $K - m$ equispaced intervals, where m is a positive integer such that

$$\tau = m(2T/K) \qquad (4)$$

Also note that each of the equispaced subintervals in Fig. 2 is of length

$$\Delta t = 2T/(K-m) \qquad 0 \leqslant m < K \quad (5)$$

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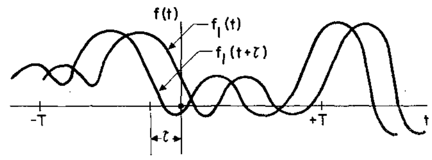


Fig. 1 Graph of $f_1(t)$ and $f_1(t+\tau)$

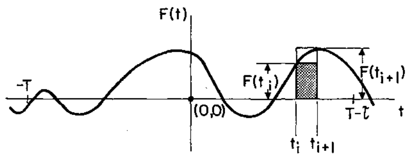


Fig. 2 Graph of $f_1(t)\,f_1(t+\tau)$

Without loss of generality, define

$$m = [K\theta] \qquad 0 \leqslant \theta < 1 \quad (6)$$

where $[K\theta]$ means the greatest integer in $K\,\theta$.

The area A_i under $F(t)$ over $t_i \leqslant t \leqslant t_{i+1}$ can be described by the inequality

$$F(t_i)\,\Delta t \leqslant A_i \leqslant F(t_{i+1})\,\Delta t \qquad (7)$$

Rearranging inequality (7) slightly and summing over $-T \leqslant t \leqslant T - \tau$ gives

$$0 \leqslant \sum_{i=0}^{K-m-1} [A_i - F(t_i)\,\Delta t] \leqslant \sum_{i=0}^{K-m-1} [F(t_{i+1}) - F(t_i)]\,\Delta t \qquad (8)$$

In inequality (8), let

$$S = \sum_{i=0}^{K-m-1} [F(t_{i+1}) - F(t_i)]$$
$$= [F(t_1) - F(t_0)] + [F(t_2) - F(t_1)] + \ldots + [F(t_{K-m-1}) - F(t_{K-m-2})] + [F(t_{K-m}) - F(t_{K-m-1})]$$

Then

$$S = F(t_{K-m}) - F(t_0) \qquad (9)$$

Substitute Eqs. (5) and (9) into inequality (8), and then introduce the factor $1/2T$ to get the following inequality:

$$0 \leqslant \frac{1}{2T} \sum_{i=0}^{K-m-1} [A_i - F(t_i)\,\Delta t] \leqslant [F(t_{K-m}) - F(-T)] \left(\frac{1}{K-m}\right) \qquad (10)$$

where $F(t_0) = F(-T)$ from Fig. 2.

Application of the fundamental theorem of integral calculus to Eq. (10) will yield an integral expression for $A_{11}(\tau)$ on $-T \leqslant t \leqslant T - \tau$. A measure of the error in the approximation of $A_{11}(\tau)$ is given by inequality (10). Thus

$$\epsilon = |[F(t_{K-m}) - F(-T)][1/(K-m)]| \qquad (11)$$

Let c be the maximum value of $|[F(t_{K-m}) - F(-T)]|$ for all $0 \leqslant m < K$, and then

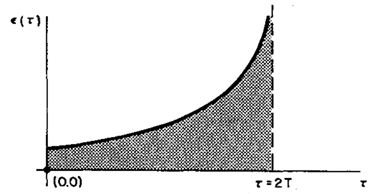
$$\epsilon \leqslant c/(K-m) \qquad (12)$$

From Eq. (4)

$$\epsilon(\tau) \leqslant c/[K(1 - \tau/2T)] \qquad (13)$$

Inequality (13) defines an error region as shown in Fig. 3.

Fig. 3 Graph of error ϵ vs time lag τ



From inequality (13), it is evident that for $\tau_1 < \tau_2$

$$\epsilon(\tau_1) < \epsilon(\tau_2) \tag{14}$$

Now if τ_1 and τ_2 are the maximum values of τ in two separate autocorrelation computations for the same $f_1(t)$, inequality (14) states that the greater the maximum time lag, the more error is introduced into the final results.

Numerical Example

In conjunction with inequalities (13) and (14), the effect of increasing the maximum time lag τ now will be demonstrated. Thus, let τ_1 be 10% of the total time record on $-T \leqslant t \leqslant T$ and τ_2 be 50% of the same time record. From (13), for $\tau_1 = 0.2T$

$$\tfrac{9}{10}\epsilon(0.2T) \leqslant c/K$$

and for $\tau_2 = T$

$$\tfrac{1}{2}\epsilon(T) \leqslant c/K$$

From these inequalities, it follows that

$$[\tfrac{9}{10}\epsilon(0.2T)]\cdot[-2/\epsilon(T)] \leqslant -1$$

and from (14)

$$\epsilon(0.2T) < \epsilon(T) \leqslant \tfrac{9}{5}\epsilon(0.2T) \tag{15}$$

Since the error introduced by letting the maximum time lag τ be $\frac{1}{2}$ the data record can be almost twice the error introduced by letting the maximum time lag τ be $\frac{1}{10}$ the data record, it is concluded that the maximum lag in digitally computing the autocorrelation function should not exceed 10% of the total data record.

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Orbit-Resonance of Satellites in Librational Motion

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Introduction

A GRAVITATIONALLY oriented satellite executes free rotational oscillations about its mass center at either of two distinct frequencies determined by the mass distribution.[1,2] Since both frequencies are not appreciably greater than the orbital frequency, these are also comparable to the natural frequency of an orbital perturbation. A calculation is presented which shows that the two types of motion, although not dynamically coupled, nevertheless do interact. It is demonstrated that orbital perturbations serve to excite the low-frequency mode of rotary oscillation and that, for orbits of small eccentricity, this occurs at a near-resonant condition.

Stability of Satellite Orbits

Circular orbits are shown in treatises on dynamics to be stable for a class of inverse-power attraction laws which includes Newtonian gravitation. It is shown here first that periodic oscillation about the basic orbit also occurs for noncircular orbits in an inverse-square force field, by considering a small perturbation on an arbitrary "undisturbed" orbit. If $R_0(t)$ and $\vartheta(t)$ denote polar coordinates that locate the mass center of the satellite in its basic undisturbed orbit, and $r'(t)$, $\theta'(t)$ represent the corresponding perturbation quantities, then the "linearized" equations of perturbed motion are

$$\ddot{r}' - R_0\dot{\vartheta}\dot{\theta}' + \dot{\vartheta}^2 r' = (2G/R_0^3)r' \tag{1}$$

and

$$(2\dot{\vartheta}r' + R_0\dot{\theta}')R_0 = \eta \tag{2}$$

where R_0 and ϑ satisfy the equations for the basic orbit, dots denote time differentiation, G is the constant of earth gravitation, and η is an integration constant. Powers and products of disturbance quantities have been neglected systematically in Eqs. (1) and (2), so that these form a system of linear ordinary differential equations. These govern motion of satellite mass center and thus are unaffected by rotary oscillations about that point. Hence they can be analyzed by themselves, and the characteristics of the motion are determined by eliminating the angular variable θ', leaving

$$\ddot{r}' + [2\dot{\vartheta}^2 + (2E/R_0) - (\dot{R}_0^2/R_0^2)]r' = \dot{\vartheta}\eta/R_0 \tag{3}$$

Only for circular orbits are the coefficient and right-hand side both constants, and evaluation of the total energy of the motion E then leads to the equation of the harmonic oscillator at frequency equal to the orbital frequency. Thus, orbital perturbations occur at precisely the rate of 1 cycle/orbit for circular orbits. In the more general case, the coefficients are not constant and the motion is not simple harmonic, but the form of the equation shows that, for orbits of small eccentricity, the "instantaneous" frequencies of orbital perturbations

$$\omega_0 = [2\dot{\vartheta}^2 + (2E/R_0 - \dot{R}_0^2/R_0^2)]^{1/2} \tag{4}$$

are not much different from 1 cycle/orbit. Equations (1) and (2) indicate a 90° phase lag between r' and θ', which will be seen below to have a counterpart in the rotary motions. For simplicity, only circular orbits will be considered henceforth.

Rotational Oscillations about Mass Center (Librations)

Gravity-gradient satellite dynamic characteristics are examined by computing the total moment of momentum of the satellite with respect to its mass center and relating this to the resultant torque moment of gravitational forces acting on constituent mass particles of the body. Orbital oscillations of the type just considered entail angular perturbations θ' that must be included in calculation of moment of momentum. The equation in vector form

$$d\mathbf{h}_0/dt = \mathbf{M}_0 \tag{5}$$

is evaluated for small angular displacements α,β,γ with respect to principal inertia axes, these being shifted only slightly from equilibrium orientation in space. In equilib-

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